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ALGEBRAIC COHOMOLOGY OF TOPOLOGICAL GROUPS

BY

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ABSTRACT. A general cohomology theory for topological groups is described, and shown to coincide with the theories of C. C. Moore [12] and other authors. We also recover some invariants from algebraic topology.

This article contains proofs of results announced in [15]. We consider algebraic cohomology groups of topological groups, which are shown to include the invariants considered by Van Est [6], Hochschild and Mostow [7], C. C. Moore [12], and Tate (see [5]). We identify some of these groups as invariants familiar from algebraic topology.

Let G be a topological group. A topological G-module is an abelian topological group A together with a continuous map $G \times A \to A$ satisfying the usual relations $g(a + a') = ga + ga'$, $(gg')a = g(g'a)$, $1a = a$. The category of topological G-modules and equivariant continuous homomorphisms is a quasi-abelian category in the sense of Yoneda [16], and hence we get Ext functors just as in an abelian category. A proper short exact sequence will be a sequence $0 \to A \to B \xrightarrow{u} C \to 0$ of topological G-modules which is exact as a sequence of abstract groups and such that A has the subspace topology induced by its embedding in B, and such that u be an open map. For any G-module A we define the algebraic cohomology groups $H^i(G, A)$ to be the ith Ext group $\mathrm{Ext}^i(Z, A)$, where Z denotes the group of integers with the discrete topology and trivial G-action.

There is another set of short exact sequences we might have chosen which also give the category of topological G-modules the structure of a quasi-abelian S-category in the sense of Yoneda. We might have demanded that in addition to being exact in the previous sense, there be a continuous map $s: C \to B$ such that the composition $u \circ s$ be the identity on C. If G is locally compact we recover the "continuous cochains" theory, which is discussed in [5], [6], and [7]. If G is not locally compact it must be shown that continuous cochains are effaceable, i.e. that for any continuous cocycle $c: G^n \to A$ there is a short exact sequence $0 \to A \xrightarrow{r} B \to C \to 0$ such that $r \circ c$ is the coboundary of a

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continuous cochain $c': G^{n-1} \to B$. If G has the weak topology with respect to a countable collection of compact sets, this will follow from a lemma of Milnor [11].

In this paper we consider only complete metric G-modules. This is made plausible by a theorem of L. Brown, [2] that if C and A are complete metric G-modules, then the groups $\mathrm{Ext}^n(C, A)$ do not depend on whether we consider all, all pseudometrizable, or all complete metric G-modules, provided that G is weakly separable (i.e. that any uniform cover of G has a countable subcover). Furthermore our arguments also apply to the category of complete separable metric G-modules, hence to the functors of [12].

1. **Definition of the** $H^i(G, A)$. (See [16], also [9, Chapter 12, 5].) Let M be an additive category (with direct sums) and $\phi: A \to B$ be a map in M. A map $N \to A$ is called the kernel of ϕ if the induced sequence of abelian groups $0 \to \mathrm{Hom}(C, N) \to \mathrm{Hom}(C, A) \to \mathrm{Hom}(C, B)$ is exact for any object C of M. Dually a map $B \to L$ is called the cokernel of ϕ if the sequence

$$0 \to \mathrm{Hom}(L, C) \to \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$$

is exact for any object C of M. This implies that the compositions $N \to A \to B$ and $A \to B \to L$ are 0.

Definition. A sequence $0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0$ of maps in M is called proper exact if σ is the kernel of τ and τ is the cokernel of σ. An n-term long exact sequence in M is a sequence of short exact sequences

$$S_i = 0 \to A_i \xrightarrow{\sigma_i} B_i \xrightarrow{\tau_i} C_i \to 0, \qquad 1 \le i \le n,$$

such that $C_i = A_{i+1}$ for $1 \le i < n$. It will usually be written

$$0 \to A_1 \xrightarrow{\sigma_1} B_1 \xrightarrow{\rho_1} B_2 \cdots \xrightarrow{\rho_{n-1}} B_n \xrightarrow{\tau_n} C_n \to 0$$

where $\rho_i = \sigma_{i+1} \circ \tau_i$. Yoneda defines $\mathrm{EXT}^n(C, A)$ as the set of n-term long exact sequences with $A_1 = A$, $C_n = C$.

Definition (Yoneda). An additive category is called quasi-abelian if it satisfies the following conditions (Q) and (Q*):

(Q) Any proper exact sequences $0 \to A \to B' \to C' \to 0$ and $0 \to C \to C'$ can be combined into a commutative diagram with proper exact rows and columns:

(Diagram Q)

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & B & \to & C & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & B' & \to & C' & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & D & = & D & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

(Q*) Any proper exact sequences $0 \to A \to B \to C \to 0$ and $A \to A' \to 0$ can be combined into a commutative diagram with proper exact rows and columns:

(Diagram Q*)
$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & D & = & D & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & A' & \to & B' & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

A quasi-abelian S-category is an additive category with a distinguished subset S of proper exact sequences which satisfy Q and Q^*.

As an example we have the category of all abelian topological groups and all proper maps thereof; in this case a map is proper if and only if it is open with respect to the relative topology of its range. In Diagram Q, C is a closed subgroup of C', B is its inverse image in B' which is again a closed subgroup. Since $B \supset A$ we have $B/B' \cong C/C'$ which is D. This verifies (Q). In Diagram Q^*, D is the kernel of $A \to A'$, $B' \cong B/D$, and $A \supset D$, we have $B'/A' \cong B/A \cong C$. Also for any fixed Hausdorff topological group G one can consider the category $\mathfrak{M}_G$ of G-modules, complete metrizable abelian topological groups A with continuous action $G \times A \to A$ satisfying $1a = a$, $(gg')a = g(g'a)$ and $g(a + a') = ga + ga'$ and continuous equivariant homomorphisms. As with abelian topological groups the totality of all proper maps gives $\mathfrak{M}_G$ the structure of a quasi-abelian S-category and henceforth $\mathfrak{M}_G$ will be assumed to be equipped with this structure. In a quasi-abelian category Yoneda defines functors $\mathrm{Ext}^n(C, A)$ as a certain quotient of $\mathrm{EXT}^n(C, A)$, the set of n-term long exact sequences. Let $0 \to A \to B_1 \to \cdots \to B_n \to C \to 0$ and $0 \to A \to B_1' \to \cdots \to B_n' \to C \to 0$ be elements of $\mathrm{EXT}^n(C, A)$. We say there is a map between them if there exists a commutative diagram

$$\begin{array}{ccccccccccccc} 0 & \to & A & \to & B_1 & \to & \cdots & \to & B_n & \to & C & \to & 0 \\ & & \| & & \downarrow & & & & \downarrow & & \| & & \\ 0 & \to & A & \to & B_1' & \to & \cdots & \to & B_n' & \to & C & \to & 0 \end{array}$$

$\mathrm{Ext}^n(C, A)$ is defined as the quotient of $\mathrm{EXT}^n(C, A)$ under the equivalence relation generated by maps between long exact sequences.

If A is a G-module, we define $H^i(G, A)$ to be $\mathrm{Ext}^i_{\mathfrak{M}_G}(Z, A)$, where Z is the group of integers with the discrete topology and trivial G-action.

It follows from Yoneda's work that if $0 \to A \to B \to C \to 0$ is a proper exact sequence of topological G-modules, we have a long exact sequence

$$0 \to H^0(G,A) \to H^0(G,B) \to H^0(G,C) \to H^1(G,A)$$
$$\to H^1(G,B) \to H^1(G,C) \to H^2(G,A) \to \cdots .$$

We can then complete a diagram chase to show the $H^i(G, A)$ are universal functors [4] and prove a "Buchsbaum criterion" for the $H^i(G, A)$. Namely an exact connected sequence of functors $\tilde{H}^i(G, A)$ is naturally isomorphic to the $H^i(G, A)$ if $\tilde{H}^0(G, A) \cong H^0(G, A)$ and satisfies the following condition:

For $i > 0$ and $X \in \tilde{H}^i(A)$ there exists a proper monomorphism $\theta: A \to B$ such that $\theta_*(X) = 0$. It follows immediately from Buchsbaum's criterion and results of C. C. Moore [12] that the functors of [12] coincide with the $H^i(G, A)$ described above.

Henceforward let G be locally compact σ-compact and let $\mathfrak{M}_G$ be the category of complete metric G-modules. If A is a G-module let $C^n(G, A)$ be the set of continuous maps of the n-fold cartesian product G^n into A. Let $\delta_n: C^n(G, A) \to C^{n+1}(G, A)$ be the usual coboundary operator: $\delta_n f(g_0, \cdots, g_n) = g_0 f(g_1 \cdots g_n) - f(g_0 g_1, g_2, \cdots, g_n) + \cdots \pm f(g_0, \cdots, g_{n-1})$. Define $\tilde{H}^n(G, A)$ as the nth cohomology group of the complex $0 \to C^0(G, A) \xrightarrow{\delta_0} C^1(G, A) \xrightarrow{\delta_1} \cdots$. $C^0(G, A) \cong A$ are the continuous functions from G^0 = point into A. $\delta_0 a = ga - a$ so $\tilde{H}^0(G, A) \cong \operatorname{Hom}_{\mathfrak{M}_G}(\mathbf{Z}, A) \cong H^0(G, A)$. If $F(G, A) \in \mathfrak{M}_G$ is the module of continuous functions from G into A topologized with the compact open topology, the natural map $A \to F(G, A)$ kills $\tilde{H}^i(G, A)$ (cf. [7]). The $\tilde{H}^i$ form an exact connected sequence of functors if we demand that all short exact sequences $0 \to A \to B \xrightarrow{\pi} C \to 0$ have a section, i.e. a continuous map $\rho: C \to B$ such that $\pi \circ \rho =$ identity. We call this the "continuous cochains" theory.

Now suppose G is zero-dimensional. Then the $\tilde{H}^i(G, A)$ are exact for arbitrary short exact sequences because of the following theorem of Michael:

Theorem M. *If $\pi: B \to C$ is an open homomorphism of complete metric topological groups, and if $q: G \to C$ is a continuous map of a 0-dimensional paracompact space into C, then there exists a continuous map $\rho: G \to B$ with $\pi \circ \rho = q$.*

Hence by Buchsbaum's criterion

Theorem 1. *If G is locally compact, σ-compact, zero-dimensional, $H^i(G, A) \cong \tilde{H}^i(G, A)$ defined above.*

We now show how to embed an arbitrary complete metric G-module in a contractible complete metric G-module. Let A be a complete metric G-module with a bounded, invariant metric ρ. Let S be the topological group of step functions from the unit interval $[0, 1]$ to A which have only finitely many steps with metric obtained from integrating ρ on $[0, 1]$ and natural G action. $G \times S \to S$ is continuous since the functions of S assume only finitely many values. Let $\mathfrak{S}_A$ be the completion of S which is also a G-module by [2] or [12]. $\mathfrak{S}_A$ will be the space measurable functions $[0, 1] \to A$ modulo functions almost everywhere 0. Let C: $\mathfrak{S}_A \times [0, 1]$ be defined by

$$C(f, \alpha)(x) = 0, \quad \text{if } x < \alpha,$$
$$= f(x), \quad \text{if } x \geq \alpha.$$

C is a contraction of $\mathfrak{S}_A$ which shrinks all distances; hence $\mathfrak{S}_A$ is contractible and locally contractible. In fact any contractible topological group is locally contractible.

2. Some fibration properties of open homomorphisms.

Lemma 1. *Let $0 \to A \to B \xrightarrow{\rho} C \to 0$ be an exact sequence of complete metric abelian groups with A locally arcwise connected. Let PB (respectively PC) denote the space of continuous paths in B (respectively C) starting at the identity with the topology of uniform convergence. Then the induced map $\rho_*: PB \to PC$ is open.*

Proof. Since PB and PC are complete metric abelian topological groups, it will be enough to show ρ_* almost open by the open mapping theorem. Let d be an invariant metric on B. d induces an invariant metric d' on C by taking the distance between cosets of A. Let $\epsilon > 0$; we must show there exists a δ such that for any path in C, $p: [0, 1] \to C$ such that for all $x \in [0, 1]$, $d(p(x), id) < \delta$ and for all $\gamma > 0$ there is a path in B, $q: [0, 1] \to B$ such that for all $y \in [0, 1]$, $d(q(y), id) < \epsilon$ and $d(\rho q(y), p(y)) < \gamma$. Now d induces a metric on A. Pick $\delta < \epsilon/4$ and such that any two points in A at distance $< 4\delta$ of the identity of A can be joined by a path in A, all of whose points s satisfy $d(s, id) < \epsilon/4$. Now by a theorem of Michael [11, II, Theorem 1.2], p lifts locally to $q': [0, 1] \to \{x \in B \mid d(x, id) < \delta\} = N$. Since $[0, 1]$ is compact we can assume it covered by a finite number of subintervals $I_i = [a_i, b_i]$, $i = 1, \cdots, n$ with $a_1 = 0$, $b_n = 1$, $a_i < b_{i-1}$, $b_i < a_{i+2}$ and $q_i': I_i \to N$ continuous such that $\rho \circ q_i' = p \mid I_i$. Now $d(q_i'(b_i), q_{i+1}'(b_i)) < 2\delta$ so there is a path $r_i: [0, 1/2] \to \rho^{-1}(p(b_i))$ with $r_i(0) = q_i'(b_i)$, $r_i(1/2) = q_{i+1}'(b_i)$, $d(r_i(x), id) < \epsilon$. Pick $\beta < \min_i (b_i/10, (b_{i+1} - b_i)/10)$ and such that for all i and all α with $0 \leq \alpha \leq \beta$, $d(p(b_i + \alpha), p(b_i)) < \gamma/2$.

Define q as follows: for

$$0 \leq x \leq b, \qquad q(x) = q_1'(x),$$

$$b_i + \beta \leq x \leq b_{i+1}, \qquad q(x) = q_{i+1}'(x),$$

$$b_i \leq x \leq b_i + \tfrac{1}{2}\beta, \quad q(x) = r_i((x - b_i)/\beta),$$

$$b_i + \tfrac{1}{2}\beta \leq x \leq b_i + \beta, \quad q(x) = q_{i+1}''(b_i + (2(x - b_i)/\beta - 1)\beta).$$

It is clear that q has the required properties. The idea of this construction is to splice the q_j' together without going far from the origin. This proves the lemma.

Definition. A complete metric abelian topological group A is said to have property F if for any short exact sequence of complete metric abelian topological groups $0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0$, τ has the homotopy lifting property for finite dimensional (paracompact) spaces. Dimension will be understood in the sense of Lebesgue covering dimension. $\mathfrak{M}_G^F$ will denote the category of complete metric G-modules having property F, where a sequence is exact if it is exact in $\mathfrak{M}_G$.

Proposition 1. *Let* $0 \to A \to B \to C \to 0$ *be exact in* $\mathfrak{M}_G$ *where* A, C *have property* F. *Then* B *has property* F.

Proof. Let $0 \to A \to B \to C \to 0$ in $\mathfrak{M}_G$ where A and C have property F. Let also $0 \to B \to D \xrightarrow{\rho} E \to 0$ in $\mathfrak{M}_G$. Consider the diagram in $\mathfrak{M}_G$.

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & A & & A & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & B & \to & D & \xrightarrow{\rho} & E & \to & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \tau} & & & & \\ 0 & \to & C & \to & C' & \xrightarrow[\sigma]{} & E & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Let $h: X \times I \to E$ be a homotopy of which property F would guarantee a lifting. Since C has property F, h can be lifted to C'. Since A has property F, h can be lifted to D. This proves B has property F.

Corollary. $\mathfrak{M}_G^F$ *is a quasi-abelian S-category.*

Proposition 2. *If* A *is a locally compact closed subgroup of a topological group* G *the projection* $G \to G/A$ *is a fibration.*

Proof. First suppose A compact. Let h be a homotopy of $X \times I \to G/A$ and h_1 be a lifting $X \times I \to G$. Consider the set S of pairs (A_α, h_α) where A_α is closed in A, $\pi_\alpha: G \to G/A_\alpha$, $h_\alpha: X \times I \to G/A_\alpha$, $\pi_\alpha \circ h_\alpha = h$, $\pi_\alpha \circ h_1 = h_\alpha | X \times I$. We define a partial order on S. If $A_\alpha \subset A_\beta$, $\pi: G/A_\alpha \to G/A_\beta$ and $\pi \circ h_\alpha = h_\beta$ we say $(A_\alpha, h_\alpha) > (A_\beta, h_\beta)$. If $\{(A_\gamma, h_\gamma)\}_\gamma$, I is a linearly ordered subset of S we obtain

$$\tilde{h}: X \times I \to \varprojlim_{\gamma \in I} \frac{G}{A_\gamma} = \frac{G}{\bigcap_{\gamma \in I} A_\gamma}$$

and $(\bigcap_{\gamma \in I} A_\gamma, \tilde{h})$ is an upper bound. Hence Zorn's lemma applies, and S has

(A_δ, h_δ) maximal. But if $A_\delta \neq \{1\}$, A_δ has a proper closed subgroup $A_\epsilon \neq A_\delta$ such that A_δ/A_ϵ is a Lie group. Hence $G/A_\epsilon \to G/A_\delta$ has a local section and is a fibration, hence (A_δ, h_δ) cannot have been maximal. Hence $A_\delta = \{1\}$. This shows $G \to G/A$ has a homotopy lifting property for A compact. But by the structure theorem any locally compact A has an open subgroup A' such that A' has a compact normal subgroup A'' such that A'/A'' is a Lie group. $G \to G/A''$ is a fibration. Since A'/A'' is a Lie group $G/A'' \to G/A'$ is a fibration by [14, Theorem 1]. A/A' is discrete so $G/A' \to G/A$ is even a covering space. Since $G \to G/A$ is a composite of fibrations it is a fibration.

Corollary. *A locally arcwise compact metric G-module is in* $\mathfrak{M}_G^F$.

Proposition 3. *A locally connected complete metric abelian topological group has property F.*

Proof. Let PX denote the space of base-pointed paths of X. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \to & PA & \to & P\mathcal{E}_A & \xrightarrow{\phi} & P\mathcal{E}_A/A & \to & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\chi} & & \\
0 & \to & A & \to & \mathcal{E}_A & \xrightarrow{\tau} & \mathcal{E}_A/A & \to & 0
\end{array}$$

The top row is exact by Lemma 1 and ϕ has the homotopy lifting property for finite dimensional spaces since PA is locally contractible by Michael [10, Theorem 3.4, Proposition 4.1 and Corollary 4.2]. Let Z be finite dimensional, h: $Z \times I \to \mathcal{E}_A/A$, $h'\colon Z \to \mathcal{E}_A$ with $\tau \circ h' = h \mid Z \times 0$. ψ is a fibration with contractible base so it has a section $s\colon \mathcal{E}_A \to P\mathcal{E}_A$. $\chi \circ \phi$ has the HLP for Z since both X and ϕ do, hence there exists $g\colon Z \times I \to P\mathcal{E}_A$ with $g \mid Z \times 0 = s \circ h'$, and $\chi \circ \phi \circ g = h$, $\chi \circ g$ is a lifting of h to $\mathcal{E}_A$ by the commutativity of the diagram. This shows that τ has the HLP for Z.

We form the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & A & \xrightarrow{\sigma} & B & \xrightarrow{\rho} & C & \to & 0 \\
 & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi'} & & \| & & \\
0 & \to & \mathcal{E}_A & \xrightarrow{\sigma'} & P & \xrightarrow{\rho'} & C & \to & 0 \\
 & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau'} & & & & \\
 & & \mathcal{E}_A/A & = & \mathcal{E}_A/A & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Let $h\colon X \times I \to C$, $h'\colon X \to B$ with $h' = h \mid X \times 0$ and X finite dimensional.

Since $\mathcal{E}_A$ is locally contractible, ρ' has the homotopy lifting property for finite dimensional spaces again by Theorem 3.4 of [10] so there exists $g: X \times I \to P$ with $\rho' \circ g = h$ and $g \mid X \times 0 = \phi' \circ h'$. Since $r' \circ \phi' \circ h' = 0$ there exists f: $X \times I \to \mathcal{E}_A$ with $r \circ f = g$ and $f \mid X \times 0 = 0$. Since r has the HLP for X, $r' \circ (g - \sigma' \circ f) = 0$ so the range of $g - \sigma' \circ f$ lies entirely in B. Hence $\phi'^{-1} \circ (g - \sigma' \circ f)$ is defined and lifts h as required. This proves the proposition.

Proposition 4. *If A, C are in $\mathfrak{M}_G^F$,* $\mathrm{Ext}_{\mathfrak{M}_G^F}(C, A) \cong \mathrm{Ext}_{\mathfrak{M}_G}(C, A)$.

Proof. Consider

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & & & & \\ & & \| & & \downarrow & & & & \\ 0 & \to & A & \to & \mathcal{E}_B & \to & \mathcal{E}_B/A & \to & 0 \end{array}$$

with $A \in \mathfrak{M}_G^F$ and $B \in \mathfrak{M}_G^{CM}$. $\mathcal{E}_B$ is locally arcwise connected, hence $\mathcal{E}_B/A$ is locally arcwise connected and in $\mathfrak{M}_G^F$. Hence anything which is effaceable in $\mathfrak{M}_G$ is effaceable in $\mathfrak{M}_G^F$ and Buchsbaum's criterion is verified.

3. **Double complex.** We now assign to the topological group G a semisimplicial G-space $S(G)$. $S(G)$ is a semisimplicial object in the category of topological spaces with jointly continuous action of the group G and equivariant maps. The n-simplex S_n of this semisimplicial complex was the $(n + 1)$-fold cartesian power G^{n+1} of the space underlying the group G, and the faces and degeneracies were as follows:

$$d_0 g(g_1, g_2, \cdots, g_n) = gg_1(g_2, \cdots, g_n),$$
$$d_i g(g_1, \cdots, g_n) = g(g_1, \cdots, g_{i-1} g_i, \cdots, g_n) \quad \text{for } 0 < i < n,$$
$$d_n g(g_1, \cdots, g_n) = g(g_1, \cdots, g_{n-1}),$$
$$s_i g(g_1, \cdots, g_n) = g(g_1, \cdots, g_{i-1}, 1, g_i, \cdots, g_n).$$

G acts by left multiplication on the argument outside the parenthesis.

Let A be a G-module. Using the action of G on S_n and A we form the space $S_n \times_G A$ and consider the natural projections $p_n: S_n \times_G A \to S_n/G$. The faces and degeneracies of $S(G)$ induce faces and degeneracies on the $S_n \times_G A$ and on the S_n/G making them into semisimplicial spaces and these faces and degeneracies commute with the natural projections p_n. Let T_n be the sheaf of germs of continuous sections of p_n. Since the identity of A is fixed by G, there is an isomorphism of T_n with the sheaf of germs of continuous A-valued functions on S_n/G. The T_n have faces and degeneracies induced by the faces and degeneracies of $S(G)$. The T_n thus form a semisimplicial sheaf $T(G, A)$ over the S_n/G, i.e. a semisimplicial object in the category of spaces with sheaves and cohomomorphisms. We apply the canonical semisimplicial resolution functor [1, Chapter II] to the semisimplicial sheaf $T(G, A)$. We then get a double complex of abelian groups, $D^{p,q}(G, A) = \mathfrak{F}^p(S_q/G, T_q)$ the pth stage of the canonical semisimplicial resolution of the sheaf T_q over S_q/G. We denote the pth cohomology group of this double complex by $\hat{H}^p(G, A)$.

Associated to $D^{p,q}$ is a spectral sequence with E_1 term $E_1^{p,q} \cong H^p(S_q/G, T_q)$, the sheaf cohomology of S_q/G with coefficient sheaf T_q. Since S_0/G is a point, $E_1^{0,0}$ is the abstract group underlying A. If $z \in A$, $d_1(a) \in H^0(S_1/G, T_1)$ is a continuous function from $S_1/G \cong G$ into A. In fact $d_1(a)$ maps g into $ga - a$, hence we see that $H^0(G, A) \cong A^G \cong \hat{H}^0(G, A)$ where A^G is the abstract group of points of A fixed by G.

Now suppose G is finite dimensional. G is then locally $Z \times N$ where Z is a simplex and N is 0-dimensional. Now let $0 \to A \to B \xrightarrow{r} C \to 0$ be a short exact sequence in $\mathfrak{M}_G^F$. We will show $r_*: D^{p,q}(G, B) \to D^{p,q}(G, C)$ is surjective. If $q = 1$ and l is a germ of a continuous map of G into C, l can be represented by a continuous map $l: Z \times N \to C$ where N is 0-dimensional and Z is a simplex. If $z \in Z$, $l | z \times N$ can be lifted by Theorem M. But Z is contractible hence the lifting $\overline{l}$ such that $r \circ \overline{l} = \tilde{l}$ is guaranteed by property F. Now $D^{p,q}(G, {}^*)$ is easily seen to be left exact on $\mathfrak{M}_G^F$ hence exact on $\mathfrak{M}_G^F$. We conclude that $H(G, {}^*)$ is an exact connected sequence of functors on $\mathfrak{M}_G^F$.

To prove effaceability we first consider the proper injection $A \to \mathfrak{S}_A$. Since $\mathfrak{S}_A$ is contractible we have by [4, Lemma 4] that $E_1^{p,q}(G, \mathfrak{S}_A) = 0$ for $p > 0$. Hence $\hat{H}^*(G, \mathfrak{S}_A)$ is given by the complex of continuous cochains. Since G is locally compact continuous cochains are effaceable, and it follows that continuous cochains are effaceable in $\mathfrak{M}_G^F$. We have verified Buchsbaum's criterion for the $\hat{H}^*(G, A)$. Therefore:

Theorem 2. *If G is locally compact, σ-compact, finite dimensional and A has property F, $H^*(G, A) \cong \hat{H}^*(G, A)$ described above.*

4. **Spectral sequence.** In this section all groups will be finite dimensional, locally compact, σ-compact and all modules will be in $\mathfrak{M}_G^F$.

If A is a vector space the spectral sequence collapses from E_2 onward and we get:

Theorem 3. *$H^*(G, A)$ is given by the complex of continuous cochains if A is a vector group.*

Corollary. *If G is a connected Lie group $H^*(G, A) \cong H^*(\mathfrak{G}, \mathfrak{K}, A)$ the Lie algebra cohomology of G modulo the Lie algebra of a maximal compact subgroup, if A is a finite dimensional vector space on which G acts linearly and differentiably.*

Proof. Hochchild and Mostow [7] have shown $H^*(\mathcal{G}, \mathcal{K}, A)$ is given by continuous cochains in this case.

Now let A be a discrete G-module. We will see that the algebraic cohomology $H^*(G, A)$ coincides with the sheaf cohomology of the classifying space. Let π: $E_G \to B_G$ be a principal universal G-bundle with paracompact base. There is a semisimplicial G-space whose n-simplex is the $(n+1)$-fold fiber product F_n of E_G over B_G, by regarding the $(n+1)$-fold fiber product as the set of maps of $\{0, 1, \cdots, n\}$ into E_G whose range is contained in a single G-orbit, G acts on $E_G \times_{B_G} E_G \cdots \times_{B_G} E_G$ by the diagonal action. Consider the sheaves of germs of continuous sections of the associated bundles $F_n \times_G A \to F_n/G$. They form a semisimplicial sheaf and by applying the canonical semisimplicial resolution functor we get a double complex which we denote by $R^{p,q}$. The injection of G into the fiber of π induces a homomorphism $R^{p,q} \to D^{p,q}(G, A)$. This induces a map from the first spectral sequence of the double complex $R^{p,q}$ into the spectral sequence described in the last section. On the E_1 terms we get the map:

$$\begin{array}{ccccccc} 0 \to & H^*(E_G, A) & \text{——} & H^*(E_G \times_{B_G} E_G, A) & \to \cdots \\ & \downarrow & & \downarrow & \\ 0 \to & H^*(\text{point}, A) & \to & H^*(G, A) & \to \cdots . \end{array}$$

But

$$F_n = \overbrace{E_G \times_{B_G} \cdots \times_{B_G} E_G}^{n+1 \text{ times}}$$

is homeomorphic to $E_G \times \overbrace{G \times \cdots \times G}^{n \text{ times}}$ which is homotopy equivalent to $\overbrace{G \times \cdots \times G}^{n \text{ times}}$. Therefore by the homotopy axiom for sheaf cohomology with constant coefficients [2] we have an isomorphism of E_1 terms. Hence the E_∞ terms coincide.

Now for each point x of B_G pick a section s_x: $B_G \to E_G$ which is continuous in some neighborhood of x. For an n-tuple $(e_1, \cdots, e_n)$ in $E_G \times_{B_G} \cdots \times_{B_G} E_G$ with $\pi(e_i) = b$ define k_x: $F_n \to F_{n+1}$ by $k_x(e_1, \cdots, e_n) = (s_x(b), e_1, \cdots, e_n)$. Now an element of $R^{p,q}$ is represented by a function f: $(F_q)^{p+1} \to A$ so define h: $R^{p,q} \to R^{p,q-1}$ by $hf(X_0, \cdots, X_p) = f(k_b(X_0), k_b(X_1), \cdots, k_b(X_p))$ where $b = \pi(X_0)$. h is well-defined since s_b is continuous in a neighborhood of b. Let d: $R^{p,q} \to R^{p,q+1}$ be induced by the space map. d is then the 0th differential of the second spectral sequence of the double complex $R^{p,q}$. $dh + hd =$ identity unless $q = 0$. The kernel of d on $R^{p,0}$ consists just of functions constant on the G-orbits of E_G. Hence the E_1 term of the second spectral sequence of $R^{p,q}$ is the canonical resolution of the locally constant sheaf A on B_G. Therefore

Theorem 4. *$H^*(G, A)$ is the sheaf cohomology of the classifying space B_G with coefficients in the locally constant sheaf A, if A is a discrete G-module.*

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